

## DUALITY AND INDETERMINACY PRINCIPLE IN STRING THEORY

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## ABSTRACT

We give an elementary explanation about how string theories overcome the ultraviolet difficulty of the local field theories. The indeterminacy principle is reinterpreted as a limitation on the smallness of the domain of observations.

## 1. INTRODUCTION

One of the most attractive properties of the string theory<sup>1</sup> as a candidate for the fundamental unified theory of nature including gravity is that it resolves the renowned ultraviolet difficulty which is inherent in local quantum field theories. How the ultraviolet problem is circumvented is seen at least mathematically by examining the singularities<sup>2</sup> of the integrand of the string-loop amplitudes expressed in the moduli space of Riemann surfaces with handles and/or holes. The singularities can occur only on the boundary of the moduli space at which a Riemann surface with a definite topology is connected with another Riemann surface with a different topology. The local neighbourhood of a limiting Riemann surface where the topology change occurs is always conformally equivalent with an infinitely long cylinder (or long belt depending on situations), which is expected to be dominantly mapped into long cylindrical (or belt-like) world sheets in spacetime. From this point of view, the single most crucial property for the elimination of the ultraviolet difficulty seems to be

the conformal invariance.

Physically, however, the above explanation does not seem sufficiently convincing mainly because it is not expressed in a direct spacetime picture. Therefore it is not completely valueless to seek after the ways for understanding how strings see the short distance structure of spacetime and to complement the results of the string-loop calculation with much more elementary and intuitive languages. In this note I would like to try to do this and to make further related remarks.

## 2. DUALITY AS AN INDETERMINACY PRINCIPLE

The source of the ultraviolet divergences in local field theories can be traced back to Heisenberg's indeterminacy principle. Consider the time-energy relation

$$\Delta t \cdot \Delta E \geq \hbar \quad (1)$$

This relation originally means that any observation performed to a quantum system within a time interval  $\Delta t$  induces an uncertainty  $\Delta E$  of energy. More precisely,  $\Delta t$  is taken to be the time interval between two sequential observations and  $\Delta E$  is the fluctuation of the difference of the energies obtained. In local field theories,  $\Delta t$  can be taken to be the time interval between the interaction vertices. As  $\Delta t \rightarrow 0$ , (1) implies that quantum fluctuations with arbitrarily large energy (and hence large spatial momenta)  $\Delta E \sim \hbar(\Delta t)^{-1}$  can contribute to physical amplitudes. The divergence is caused by the fact that the number of particle states increases as  $\Delta t^{1-D}$  where  $D$  is the spacetime dimensions. It has been recognized through numerous unsuccessful attempts that this ultraviolet difficulty cannot be remedied within the framework of local field theory especially when the gravitational interaction is taken account.

A basic and traditional way out of this problem was to abandon the locality of the interactions. Sufficiently high momenta would then be cut off. In fact, if the quantum string is viewed as a collection of infinite number of local fields which are obtained by

making the normal mode expansion to diagonalize the mass operator, the absence of ultraviolet divergence can partially be understood as due to the violation of locality. For example, the general vertex which represents the connection condition of string configurations takes the Gaussian form with respect to the 'center-of-mass' momentum variables. A consequence of this is that the high momentum behavior of the string amplitudes is softened to the Regge-pole behavior replacing the power-like behavior of local field theories. Unfortunately, it is very difficult to formulate the string theory based on such a non-local picture. We know that the string theory can be formulated within the framework of quantum mechanics in which causality and unitarity is satisfied in the same sense as in local gauge field theories. (Remember the light-cone gauge formulation.)

Then the following question comes to mind: If the center-of-mass momenta are cut off owing to the non-locality as explained above, how should the relation (1) be interpreted? Roughly, the answer is the following. In string theory, there is an infinite tower of states with arbitrarily large masses. Hence, energy can become arbitrarily large without acquiring large spatial momenta. Now the intrinsic spatial extension of the string states becomes larger as the mass increases. Thus, in the string theory, the large energy fluctuation actually means large fluctuation with respect to the intrinsic spatial extension of string. Led by this observation, we propose to reinterpret the relation (1) as an indeterminacy relation between the interval of observations and the fluctuations with respect to the intrinsic extension of the 'observations':

$$[\text{interval}] \cdot [\text{fluctuation of extension}] \geq \lambda^2 \quad (2)$$

where  $\lambda$  is the universal length parameter given by  $(4\pi\alpha' \hbar)^{1/2}$  where  $\alpha'$  being the slope parameter. (One unit is such that  $c=1$ .)

Here an 'observation' is a synonym of a string interaction.

(2) implies that there is a limitation about the smallness of the spacetime domain where arbitrary possible observation in string theory is performed. In this sense there is no room for ultraviolet

difficulty in string theory.\*)

To make (2) more precise, we have to give a definition of the intrinsic extension of a string. Take for definiteness the case of closed string in the light-cone gauge. The transverse coordinates are represented by the normal mode expansion ( $0 \leq \sigma \leq 2\pi$ ,  $i=1, \dots, D-2$ ,  $D=26$ )

$$x^i(\sigma) = \frac{1}{\sqrt{\pi}} [x_0^i + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^i e^{in\sigma} + \tilde{\alpha}_n^i e^{-in\sigma})] \quad (3)$$

The set of mass eigenstates coincides with the occupation number basis of the Fock space with respect to the canonical commutation relations

$$[\alpha_n^i, \alpha_m^j] = n\lambda^2 \delta_{n+m,0} \delta^{ij} = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j]$$

$$[\alpha_n^i, \tilde{\alpha}_m^j] = 0, \quad \alpha_n^i |0\rangle = 0 = \tilde{\alpha}_n^i |0\rangle \quad (n > 0). \quad (4)$$

There is a trouble in defining the extension of a string configuration. One of the most natural measure for the fluctuation of the intrinsic extension might seem to be the following quantity,

$$[\text{Max}_{(\sigma_1, \sigma_2)} \langle (x(\sigma_1) - x(\sigma_2))^2 \rangle]^{1/2} \quad (5)$$

where  $\text{Max}_{(\sigma_1, \sigma_2)}$  indicates to take the maximum value among the choices of the  $(\sigma_1, \sigma_2)$  pair  $(\sigma_1, \sigma_2)$ . Unless  $\sigma_1 = \sigma_2$ , however,  $\langle (x(\sigma_1) - x(\sigma_2))^2 \rangle$  is always divergent for arbitrary normalizable states in the Fock space. This is due to the fact that the string-coordinate operator  $x^i(\sigma)$  at a point  $\sigma$  has no well-defined meaning for the same reason as is the case of a field operator defined at a spacetime point in ordinary local field theories. Some sort of smearing is necessary. Let us therefore introduce a set  $\mathcal{J}$  of smearing functions

\*) Similar consideration applies to any extended object provided that the intrinsic extension can increase with energy without limit. However, except for string, inclusion of gravity seems difficult.

as a complete set of orthonormalized functions on the  $\sigma$ -parameter space.

$$\mathcal{J} = \{f_n(\sigma); n \in \mathbb{Z}, \int_0^{2\pi} d\sigma f_n(\sigma) f_m(\sigma) = \delta_{nm}\} \quad (6)$$

For example, we can use

$$\begin{aligned} f_n(\sigma) &= \frac{1}{\sqrt{\pi}} \cos n\sigma & n \geq 1, \\ f_{-n}(\sigma) &= \frac{1}{\sqrt{\pi}} \sin n\sigma & n \geq 1, \\ f_0(\sigma) &= \frac{1}{\sqrt{2\pi}}. \end{aligned} \quad (7)$$

Then, define the fluctuation of the intrinsic extension of a normalized string state  $|\psi\rangle$  by

$$[\text{Max}_{(m,n)} \langle \psi | (x(f_n) - x(f_m))^2 | \psi \rangle]^{1/2} = \Delta l(\psi) \quad (8)$$

where

$$x^i(f_n) = \int_0^{2\pi} d\sigma f_n(\sigma) (x^i(\sigma) - \frac{1}{\sqrt{\pi}} x_0^i).$$

Since the singularity of the product  $x^i(\sigma_1)x^i(\sigma_2)$  is logarithmic ( $\log|\sigma_1 - \sigma_2|$ ), the quantity  $\langle 0 | (x(f_n) - x(f_m))^2 | 0 \rangle$  is uniformly bounded by a constant independent of  $n$  and  $m$  provided that  $|f_n| < C$  for arbitrary  $n$ . Hence, (8) is well defined.

We can now check the relation (2). If the interval of observations is of order  $\Delta t$ , (1) implies that the states of masses of the order  $\hbar(\Delta t)^{-1}$  are excited:

$$\frac{\hbar^2}{\lambda^4} \sum_{n=1}^{\infty} \langle (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \rangle \sim \hbar^2 (\Delta t)^{-2}.$$

The maximally extended mass eigenstate in this range is  $|\psi\rangle = (N!)^{-1} (\alpha_{-1}^i)^N (\tilde{\alpha}_{-1}^j)^N |0\rangle$  with  $N \sim \lambda^2 (\Delta t)^{-2}$ . We find

$$[\langle \psi | (x(f_1) - x(f_0))^2 | \psi \rangle]^{1/2} = \sqrt{(2N+1)\lambda^2} \sim \frac{\lambda^2}{\Delta t} \quad (9)$$

for the choice (7) and  $n=1, m=0$ . For other combinations of  $(n,m)$  the extension does not exceed (9). This implies the relation (2). It seems fairly obvious that the result does not depend on the choice of the set  $\mathcal{J}$ , although we do not try to make the argument rigorous.

Since the indeterminacy relation (2) is expressed directly using the product of a 'time-like' length and a 'space-like' length, it naturally fits in the original concept of duality, or perhaps more appropriately in the concept of 'reciprocity'<sup>3</sup>.

It should also be remarked that while (2) explains the absence of the ultraviolet problem, it in turn suggests the existence of a danger of infrared difficulty because of a large fluctuation in the long distance regime. The necessity of supersymmetry as a consistency condition on string theory arises at this point.

### 3. PROPERTIES OF STRING PROPAGATOR

To get a further confirmation of the relation (2), it is useful to study the string propagator. First let us remind ourselves the case of point particle. The point-particle propagator is in Euclidean metric

$$\Delta(x-y) = \text{const.} \int_0^{\infty} \frac{d\tau}{\tau^{D/2}} e^{-\frac{(x-y)^2}{2\tau} - \frac{\tau}{2} m^2} \quad (10)$$

In (10), the Schwinger parameter  $\tau$  characterizes the magnitude of the interval  $\sqrt{(x-y)^2}$  between the interaction points.

The closed-string propagator corresponding to (10) is, in the light-cone gauge, given by

$$\Delta[x(\sigma), y(\sigma)] = \text{const.} \int_0^{\infty} \frac{d\tau}{\tau^{13}} e^{\frac{4\tau}{\lambda^2} \sum_{n=1}^{\infty} (1 - e^{-4n\frac{\tau}{\lambda^2}})^{-24} \exp - S_{cl}}, \quad (11)$$

$$S_{cl} = \frac{1}{2\tau}(x_0 - y_0)^2 + \frac{\pi}{\lambda^2} \sum_{n=1}^{\infty} \frac{n}{\sinh \frac{2n\tau}{\lambda^2}} [ (|\mathbf{x}_n|^2 + |\mathbf{y}_n|^2) \cosh \frac{2n\tau}{\lambda^2} - 2\text{Re}(\mathbf{x}_n \cdot \mathbf{y}_{-n}) ] \quad (12)$$

where

$$\begin{aligned} \mathbf{x}^i(\sigma) &= \frac{1}{\sqrt{\pi}} x_0^i + \sum_{n \neq 0} \mathbf{x}_n^i e^{-in\sigma}, \\ \mathbf{y}^i(\sigma) &= \frac{1}{\sqrt{\pi}} y_0^i + \sum_{n \neq 0} \mathbf{y}_n^i e^{-in\sigma}. \end{aligned}$$

In (12), the zero-mode part  $(x_0 - y_0)^2$  contains all  $D = 26$  components while the non zero-modes contain only transverse components. As in (10), the moduli parameter  $\tau$  characterizes the interval  $(\sqrt{\tau} \sim \sqrt{(x_0 - y_0)^2})$  between the 'center-of-mass' coordinates of the initial and final string configurations. When  $\tau$  becomes large,  $S_{cl}$  is approximated by

$$S_{cl} \approx \frac{1}{2\tau}(x_0 - y_0)^2 + \frac{\pi}{\lambda^2} \sum_{n=1}^{\infty} n (|\mathbf{x}_n|^2 + |\mathbf{y}_n|^2). \quad (13)$$

As expected, (13) means that the propagator reduces in this limit to the sum of the particle propagators of the tachyon and massless states whose stringy extension is of order  $\lambda$ .

On the other hand, in the limit of small  $\tau$ , we find that the contribution in the second term of (12) for  $n$  satisfying  $2n\tau/\lambda^2 \ll 1$  is

$$\left( \frac{1}{2n\tau} - \frac{n\tau}{3\lambda^4} \right) |\mathbf{x}_n - \mathbf{y}_n|^2 + \frac{n\tau}{\lambda^4} (|\mathbf{x}_n|^2 + |\mathbf{y}_n|^2) + ((n\tau)^3). \quad (14)$$

This shows that the intrinsic extension, defined by (8), of the dominantly propagating states is of order  $\lambda^2/\sqrt{\tau}$  as required by (2).

The conclusion drawn from this simple calculation is that (2) is satisfied in the whole range of the moduli parameter. Although our calculations are performed for the closed bosonic string, it is clear

that the same conclusion follows for other cases as well.

We note that the form of the propagator (11) and (12) is actually valid<sup>4</sup> for any initial and final configurations which do not lie in the light-like hyperplanes, provided that the initial and final configurations are already parametrized. Only change is that the scalar products in (12) for the non-zero modes are replaced by the full  $D$ -dimensional products. In the general case, however, the manner how the propagator appears in the amplitude needs *not* be the same as in the usual Feynman rules of local field theories. Remember the well known prescription on the integration over the fundamental region of the moduli space in the case of loop amplitudes. It is important to realize that the propagator of the type (11) can be used only when the summation over the initial and final configurations is suitably restricted so as to be consistent with unitarity. At present, the only workable case is the light-cone formulation in which the configurations are restricted to the light-like hyper planes. In principle, however, there should exist definite prescriptions for any choice of the slice of space-like hypersurfaces.

As an example of a more general choice of the slice, let us consider the time-like gauge  $\partial x^0(\sigma)/\partial\sigma = 0$ . Then the scalar product  $\mathbf{x}_n \cdot \mathbf{y}_n$  in (12) is  $(D-1=25)$  dimensional. Our conclusion about the relation (2) is of course valid in this case as well. However, a small puzzle arises: Since the power 24 of the infinite-product term in (11) is smaller than 25, the pole residue is not in general positive definite. On using the same trick as in the covariant case<sup>4</sup>, we find the operator form for the propagator

$$\langle x_0, \{\mathbf{x}_n\} | \sum_{m=-\infty}^{\infty} (-)^m \frac{1}{P^2 + M^2 + \frac{\hbar^2}{\lambda^2} (3m^2 + m)} | y_0, \{\mathbf{y}_n\} \rangle \quad (15)$$

$$M^2 = \frac{\hbar^2}{\lambda^4} \left[ \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) - 2\lambda^2 \right]. \quad (16)$$

This problem is resolved by noticing that (15) is well defined only if the string configuration is parametrized. In the light-cone gauge, the parametrization was fixed by the requirement  $\partial P^+(\sigma)/\partial\sigma = 0$ .

In the time-like gauge, unfortunately, there is no preferred parametrization. Then to preserve unitarity, we have to demand that the states multiplied to (15) be parametrization independent. This means that the initial and final states,  $|\psi_1\rangle$  and  $|\psi_2\rangle$  respectively, should satisfy

$$\kappa'(\sigma) \cdot p(\sigma) |\psi_i\rangle = 0 \quad (i = 1, 2). \quad (17)$$

As a first nontrivial check of the positivity, we study the massless pole. The residue of the massless pole is given by

$$\text{res.} = \langle \psi_2 | \left\{ \sum_{i,j=1}^{D-1} \alpha_{-1}^i \tilde{\alpha}_{-1}^j | 0 \rangle \frac{1}{\lambda^4} \langle 0 | \alpha_{-1}^i \alpha_{-1}^j - | 0 \rangle \langle 0 | \right\} | \psi_1 \rangle. \quad (18)$$

The only non zero spatial component of the zero-mode momentum can be assumed to be the  $(D-1=25)$ -th component denoted by  $k$ . Then, using the Fourier components  $R_n$  defined by

$$\kappa'(\sigma) \cdot p(\sigma) = \frac{1}{4\pi} \sum_n R_n e^{-in\sigma}, \quad (19)$$

we find

$$\frac{1}{\lambda^2} \langle 0 | (\alpha_1)^{D-1} (\tilde{\alpha}_1)^{D-1} = -\frac{1}{4k^2} \langle 0 | R_1 R_{-1} + \langle 0 | \quad (20)$$

$$\frac{1}{\lambda} \langle 0 | (\alpha_1)^{D-1} = \frac{1}{2k} \langle 0 | R_1, \quad \frac{1}{\lambda} \langle 0 | (\tilde{\alpha}_1)^{D-1} = \frac{1}{2k} \langle 0 | R_{-1}.$$

By the condition (17), (20) leads to

$$\text{res.} = \langle \psi_2 | \sum_{i,j=1}^{24} \alpha_{-1}^i \tilde{\alpha}_{-1}^j | 0 \rangle \frac{1}{\lambda^4} \langle 0 | \alpha_{-1}^i \tilde{\alpha}_{-1}^j | \psi_1 \rangle. \quad (21)$$

It is desirable to find a general proof of positivity along this line,

and more ambitiously, to find a general formalism in which the arbitrariness of the choice of the space-like hypersurface is manifest in analogy with, e.g., Wheeler-DeWitt's Hamiltonian approach to quantum gravity.

#### 4. CONCLUSION

In this note, I have tried to understand the short distance property of the string theory from an elementary point of view. I have suggested to reinterpret the indeterminacy principle in string theory as a manifestation of the fact that the string theory contains a mechanism for cutting off the shorter distance contributions than the Planck length. It is my hope that these considerations, although yet very much primitive, might help our search for hidden geometrical foundation of the string theory.

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